Invited paper

Characteristics of travelling waves along the non-linear transmission lines for monolithic integrated circuits: a review

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A general analysis of non-linear wave propagation along transmission lines with voltage-dependent capacitance is presented. In particular, slow-wave structures like MIS and Schottky-barrier strip lines are examined. A spatial periodicity is included explicitly. The theoretical treatment is based on suitable equivalent circuits leading to characteristic wave equations. With regard to practical devices, the solutions show a variety of different phenomena as determined by the parameters of the non-linearity, dispersion and dissipation and the boundary conditions. Experimental results performed on a slow-wave model line are included.

1. Introduction

Monolithic microwave integrated circuits (MMICs) using Si or GaAs technologies are being considered as promising candidates for a number of applications (Pucel 1981). In particular, higher reliability and reproducibility are expected as well as smaller size and larger bandwidth. The key problem in the analysis of corresponding devices is due to a better understanding of wave propagation along various transmission lines on semiconductor substrates. This has motivated the need for a general analysis of microstrip lines and coplanar waveguides on layered substrates in recent years (Shih and Itoh 1982, Fukuoka et al. 1983, Sorrentino et al. 1984).

The basic structures which have been discussed in the earlier literature are the metal–insulator–semiconductor (MIS) microstrip line (Guckel et al. 1967, Hasegawa et al. 1971, Jaffe 1972) and the Schottky-contact microstrip line where the insulating region arises from the depletion layer of the Schottky barrier (Hughes and White 1972, Jäger and Rabus 1973, Hughes and White 1975, Jäger 1976). In the following, the MIS and Schottky coplanar striplines have been introduced (Hasegawa and Okizaki 1977) and different theoretical approaches have been applied (Aubourg et al. 1983, Fukuoka et al. 1983, Seguinot et al. 1983, Sorrentino et al. 1984). Very recently, Fukuoka and Itoh (1983 a–c) proposed a novel periodic coplanar waveguide in order to overcome the large attenuation factors of the homogeneous strip lines.

MIS and Schottky-contact transmission lines have been proven to exhibit several specific advantages for potential applications to MMICs. First of all, the slow-wave properties of both types of strip line can help to reduce the dimensions of integrated devices, provided that a sufficiently small attenuation per guide wavelength can be achieved. On the other hand, the analysis of such transmission lines provides the opportunity to describe and understand the behaviour of high-power

Received 28 January 1985; accepted 30 January 1985.
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high-frequency devices where travelling-wave concepts have to be employed, as has recently been done by Heinrich and Hartnagel (1983) in the case of the travelling-wave FET. Moreover, since the depletion layer depends on the applied bias voltage, a new class of electronically controllable devices is imaginable (see, for example, Hughes and White (1972, 1975), Jäger and Rabus (1973), Jäger et al. (1974), Jäger (1976), Jäger and Becker (1977)). Under reverse bias, a variable phase shifter and delay line can be constructed to yield, for example, a small-size low-pass filter with a DC-tunable cutoff frequency (Becker et al. 1977). Under forward bias, the high attenuation factor gives rise to extremely large losses so that the waveguide exhibits the properties of a fast microwave switch (Fleming et al. 1979).

Until now, the main emphasis has been laid on the small-signal properties of the slow-wave structures. However, under slow-mode properties the depletion layer width is also modulated by the momentary value of the RF signal voltage. As a result, the wave propagates along a basically non-linear transmission line and the influence of the non-linearity has to be taken into account. This is particularly true in all cases of high-power operation, as for example in the travelling-wave FET. However, there is a more general need for a non-linear analysis of MIS and Schottky microstrip lines. The proposed periodic transmission line may exhibit a sufficiently low attenuation together with a high characteristic impedance, so that non-linear effects should easily be observable even at low power levels. On the other hand, the design of an integrated non-linear device for high-frequency high-power operation leads to dimensions which are no longer small as compared to the guide wavelength which may drastically be decreased under slow-mode properties. Thus travelling-wave concepts have to be applied to such a distributed non-linear device. Moreover, it is expected that the specific properties of non-linear slow waves can in turn lead to potentially novel applications in MMICs, similar to the parametrically variable devices under small-signal conditions.

Nevertheless, there have already been a few investigations on the non-linear properties, especially on second-harmonic generation along MIS and Schottky-contact lines about ten years ago (Günther and Voges 1973, Rabus 1974, Eversumrode et al. 1977). However, these studies have pointed out that a more powerful concept is required in order to enhance the small values of measured efficiency. As a result of recent progress in the understanding of non-linear wave phenomena, and due to the expected improvements of strip-line technology for MMIC applications, such a concept is now available. Accordingly, in this paper a general and also a practical analysis of non-linear slow-wave propagation is attempted, to be applicable to planar MIS and Schottky-contact strip lines. For that purpose, several theoretical and experimentally verified results are transferred from the physical literature to the special microwave system, in order to provide the engineer with a quick way to estimate the influence of non-linearity, or to design a new device using MMIC technology. In doing so, some peculiarities as well as similarities, for example with the field of non-linear optics are mentioned. The theoretical predictions are experimentally tested by using model lines for slow-wave propagation.

2. Slow-wave structures

In Fig. 1 different structures of slow-wave transmission lines for MMIC applications are shown. There are Schottky-contact and MIS microstrip lines, unbalanced and balanced coplanar waveguides. Si or GaAs technology could be used (Pucel
1981), however, the coplanar lines on semi-insulating GaAs substrate with a semi-conducting epitaxial layer offers some advantages. Obviously the cross-sections in Fig. 1 (a) resemble those of conventional Schottky or MIS diodes in sandwich technology, the strip lines thus being nothing else than distributed diodes. Consequently the Schottky coplanar line in Fig. 1 (b) can be regarded as a distributed MESFET (Heinrich and Hartnagel 1983). It is also evident that the structures in Fig. 1 (c) are distributed planar diodes with the restriction that the electrical behaviour might be symmetric (see below). This concept of a distributed device has recently been applied to an IMPATT diode (Bayraktaroglu and Shih 1983).

The non-linearity of the transmission lines in Fig. 1 arises from the variable depletion-layer width depending on a DC bias voltage to define the operating point and on the AC voltage of the propagating wave. In Figs. 1 (a) and (b) it is assumed that a reverse bias voltage is applied to the strip line and to the centre conductor, respectively. In contrast to these unbalanced structures, Fig. 1 (c) shows symmetric configurations where the upper one is that of a planar BARITT diode with two Schottky contacts. Assuming slow-wave characteristics, the electric energy is always stored within the insulating and depletion layers so that the influence of non-linearity is at a maximum, and the structures in Fig. 1 mainly differ with respect to the kind of non-linearity.

In Fig. 2 are sketched some periodic slow-wave structures which have been discussed in the literature. The device in Fig. 2 (a) has been proposed as a line with possibly smaller losses and an extended range of slow-wave properties (Fukuoka and Itoh 1983 a–c). Besides, the characteristic impedance will be larger than in the equivalent homogeneous case. The coplanar line in Fig. 2 (b) with a periodically varying width of the centre conductor is similar to the travelling-wave Schottky-contact microstrip filter discussed by Becker et al. (1977). The transmission line of Fig. 2 (c), periodically loaded with lumped elements, is basically the non-linear version of the periodic slow-wave low-loss structure as introduced by Bastida and Donzelii (1979). The semi-lumped ladder line of Fig. 2 (c) has mainly been used in the past as a simple experimental set-up for studying non-linear wave characteristics (Jäger and Tegude 1978, Jäger 1978 b), the transmission line being essentially an
Figure 2. Basic periodic line structures for non-linear slow-wave propagation. (a) Coplanar waveguide on periodically doped semiconductor (cf. Fig. 1(b)), (b) low-pass filter due to a periodic centre conductor (cf. Fig. 1(b)) and (c) transmission line periodically loaded with varactor diodes.
analogue computer. It is a common feature of the slow-wave structures in Fig. 2 that the electrical and magnetic energies are now stored in spatially different volumes provided that the slowing factor is high and the guide wavelength of interest is large as compared with the length of one section. In the first instance, the periodicity yields low-pass filter characteristics, i.e. a clear dispersion in the vicinity of the cutoff frequency. This dispersion can obviously be ignored at low frequencies where the periodic transmission line behaves as a homogeneous waveguide. Of course, higher pass bands can also be significant when the wavelength becomes comparable, or smaller than, the spatial period.

3. Equivalent circuits

In the following, a largely general analysis of non-linear slow-wave propagation is outlined in three steps. Firstly, to describe the behaviour of the most promising structure in Fig. 2(a), a periodic waveguide is considered. From this ansatz the homogeneous case can easily be obtained. In a second step, it is assumed that the two alternating regions are electrically quite different to yield a clear slow-mode propagation and that the length of one section is small enough to neglect wave propagation.† Then the equivalent circuit in Fig. 3(a) can be used to describe the basic properties of wave propagation. In a final step, this equivalent circuit with lumped elements can approximately be replaced by a circuit with distributed elements according to Fig. 3(b). In both cases, the non-linearity has been introduced by

![Figure 3. Equivalent circuits of periodic non-linear slow-wave transmission lines. (a) Discrete line represented by a T-section with lumped elements, (b) fully distributed circuit with low-pass filter characteristics and (c) to (e) T-elements of the semi-lumped line of Fig. 2(c) with biasing network and different non-linearities: (c) KdV, (d) MKdV and (e) nMKdV system (see text).](image-url)

† If the electrical length of one region becomes comparable, or larger than, the wavelength a resonator results, the behaviour of which is discussed in a greater detail in §9.
a voltage-dependent capacitance or capacitance per unit length. In Fig. 3(a) the dispersion results from the discrete nature of the ladder line, whereas in Fig. 3(b) $C''$ allows for the low-pass filter dispersion. By setting $C'' = 0$, this equivalent circuit can be used to describe the homogeneous strip lines in Fig. 1 (Guckel et al. 1967, Jäger 1976, Seguinot et al. 1983). The circuit in Fig. 3(b) allows for further analytical treatments and the circuit in Fig. 3(a) can immediately be employed in numerical computations.

Clearly, the equivalent circuits can only serve as first-order approximations. The crucial point here is that the basic mechanisms are taken into account, i.e. non-linearity, dispersion and two types of dissipation. It will be shown later that this concept already yields the desired expressions up to a high accuracy and a quantitative agreement with experiments is obtained. For that purpose, three analogue computers for non-linear slow-waves have been realized (Figs. 3(c) to (e); Perini (1980)) using high-quality varactor diodes in different connections to model the three possible types of non-linearity, the asymmetric and two symmetric ones: for example, the microstrip, the balanced coplanar line and the line in Fig. 2(a) with alternating p-type and n-type dopings (see §4). Figures 3(c) to (e) also show the connection to the DC power supply to provide the DC bias voltage $V_0$.

4. Non-linear wave propagation

To characterize the non-linear wave propagation along periodic slow-wave structures, the following partial differential equation (PDE) for the line voltage $V$ is derived from the circuit in Fig. 3(b) (Jäger 1982):

$$\frac{\partial V}{\partial \xi} = g(V) \frac{\partial V}{\partial \tau} + \kappa \frac{\partial^3 V}{\partial \tau^3} - aV + b \frac{\partial^2 V}{\partial \tau^2}$$  

(1)

where $\xi = x/u_0$ and $\tau = t - x/u_0$ are transformed coordinates and $u_0 = (LC_0)^{-1/2}$ is the small-signal phase velocity. The non-linearity is expressed by $C(V) = C_0(1 - g(V))$. $\kappa = \omega_c^{-2} = LC''$ is a measure of dispersion, $a = R'C_0$ and $b = C_0/G'$ are the lossy parameters. Equation (1) is valid if in the moving coordinate system the spatial variation of $V$ due to the four contributions on the right-hand side—non-linearity, dispersion, frequency-independent and frequency-dependent losses—is small. From (1) the propagation constant $\gamma = \alpha + j\beta$ for linear waves becomes

$$\gamma = \frac{j}{u_0} \left\{ \omega + \frac{1}{2} \kappa \omega^3 - j\left(\frac{3}{2} \alpha + \frac{1}{2} b \omega^2\right) \right\}$$  

(2)

If periodicity is ignored, $\kappa = 0$ and the well-known results of slow-wave propagation along homogeneous waveguides are obtained. For the purpose of numerical computations the following difference–differential equations are derived from the circuit in Fig. 3(a):

$$V_{k+1} - 2V_k + V_{k-1} = L \frac{d^2Q_k}{dt^2} + R \frac{dQ_k}{dt}$$  

(3)

$$\frac{dQ_k}{dt} = G(V_k - V(Q_k))$$  

(4)

where $Q_k$ denotes the charge of the $k$th capacitance with $C = dQ/dV = C_0(1 - g(V))$. $V(Q_k)$ is the inverse function. Equations (3) and (4) can be converted into eqn. (1) by
introducing a length $l$ of the section (Fig. 2(c)). The small-signal dispersion of the cascaded line in Fig. 3(a) reads

$$\Gamma = 2 \arcsinh \left( \frac{1}{2} WY \right)^{1/2} \quad (5)$$

$\Gamma$ is the complex phase shift per section and $W$ and $Y$ are, respectively, the series impedance and shunt admittance per section.

The important point is now, that (1) is a well-known wave equation in applied physics (Jäger 1982). It is the lossy generalized Korteweg de Vries (GKdV) equation. The solutions strongly depend on the relative order of the four different physical contributions which for harmonic waves can be judged from (2) and from $g(V)$. Ignoring the losses for a moment, the kind of non-linearity determines the fundamental non-linear behaviour and three different cases may be distinguished: $g(V) = \delta V$ leads to the usual KdV equation, $g(V) = \delta V^2 > 0$ and $g(V) = \delta^2 V^2 < 0$ yield the modified KdV (MKdV) and the negative modified KdV (nMKdV) equation, respectively.

The general behaviour of non-linear waves as described by (1) is as follows. Studying for example the propagation of a sinusoidal input wave, the non-linearity basically produces harmonics, a process which experiences the influence of dissipation and dispersion. Qualitatively, the parameter $a$ imposes an overall attenuation whereas $b$ yields a dissipation drastically increasing with frequency thus preventing the formation of shock waves. The dispersion, on the other hand, allows interference effects between propagating Fourier components and generated ones, thus ultimately limiting the amplitudes of harmonics.

In the following, non-linear slow-wave propagation is quantitatively investigated where the dispersion is assumed to exceed the dissipation. The reasons are that at first the periodic structures seem to be most advantageous for MMIC application. Secondly, the dispersionless case has already been studied by Günther and Voges (1973) and Rabus (1974) where, however, it has been predicted that, for example, the efficiency of SHG will only be large if a low-pass filter is employed (cf. also Everszumrode et al. (1977)). Furthermore, only by using a lowpass filter the influence of dissipation can be limited in the non-linear case since the number of harmonic waves is limited. Experiments and theoretical results have confirmed this statement. In order to be quantitative, the experimental parameters of the basic KdV model line in Fig. 2(c) are (cf. Fig. 3(a)): $L = 23.7$ nH; $R = 0.051$ $\Omega$; $G = 1.35$ $\Omega^{-1}$; $C(V) = C_0(1 + (V + V_0)^{-1}$ with $C_0 = 24$ pF and $\epsilon = 0.4$ $V^{-1}$; $l = 3$ cm. The capacitance–voltage relation is that of a hyperabrupt junction (Sze 1969). In the case of small non-linearity

$$C(V) = C_0(1 - \delta V) \quad (6)$$

where $C_0 = C_a(1 + \epsilon V_0)^{-1}$ and $\delta = \epsilon(1 + \epsilon V_0)^{-1}$. At $V_0 = 4$ V, this periodic slow-wave structure exhibits a high quality factor where the influence of losses is only of second order (Jäger 1982). Equation (6) is a result of a quadratic non-linearity in the charge–voltage relation, the cubic non-linearity as realized by the transmission lines in Figs. 3(d) and (e) yield instead

$$C(V) = C_0(1 - \delta^2 V^2) \quad (7)$$

where $\delta^2 = \delta^2/4$. In the following, non-linear wave phenomena are discussed in steps depending on the number and kind of the Fourier components involved. For the theoretical treatments the circuit of Fig. 3(b) and eqn. (1) are used throughout.
5. DC bias variation

In order to be complete, two possible applications should be mentioned which arise from a DC bias variation. To begin with, even in the small-signal case, slow-wave structures may be useful for external phase modulation according to

\[ \beta(t) = \omega (LC_0(V_0(t)))^{1/2} \]  \hspace{1cm} (8)

where the bandwidth of \( V_0(t) \) extends from DC to high frequencies. The possibility of external phase variation gives rise to a number of useful devices (see, for example, Hughes and White (1975), Jäger (1976)). For example, a pulse \( V_0(t) \) impresses a frequency chirp on the microwave. Only low voltage levels and negligible power are needed. At \( x = x_0 \) one obtains from (8) and the quadratic non-linearity (eqn. (6))

\[ \omega(t) = \omega_0 \left\{ 1 - \frac{x_0}{u_0} \left( 1 - \delta V_0 \right)^{-3/2} \delta \frac{dV_0}{dt} \right\} \]  \hspace{1cm} (9)

Here \( \omega_0 \) is the carrier frequency.

On the other hand, a DC power may also be generated by non-linear processes on the transmission line. One source of such self-biasing is the strongly non-linear conductance of the depletion layer in the forward direction. The transmission line now reveals the properties of a distributed rectifier (Jäger 1978 a). The typical waveform on a Schottky-contact microstrip line is shown in Fig. 4 using an input power of about 10 W. The efficiency of this high-power device is found to be 45% provided that an optimum angle of current flow is chosen and a maximum DC power is generated.

6. Wave mixing

In this section some special aspects concerning the propagation and interaction of harmonic waves are reviewed. The analysis starts with the ansatz of a limited
number of Fourier components in (1). As a rule, lowering the fundamental frequency results in a broader spectrum to be considered since the dispersion of the low-pass filter is of great importance. In other words, only those Fourier components with comparable phase velocities will obtain large amplitudes. Hence for practical purposes it is always advantageous to adjust the dispersion in that way that only the desired frequencies are generated. One way to achieve this by MMIC technology is discussed in the following special case of harmonic frequency generation.

The general discussion of such parametric interactions starts in the frequency domain, with the ansatz

$$V(x, t) = V_0(t) + \text{Re} \sum_{i=1}^{n} \tilde{V}_i(x, t) \exp \{j(\omega_i t - \gamma_i x)\}$$

(10)

The complex amplitudes $\tilde{V}_i$ are considered to be slowly varying functions of distance and time. $V_0(t)$ is a slowly varying bias voltage due to non-linear interactions (see §8). The $\omega_i$ denote the allowed frequencies and $\gamma_i$ the corresponding complex phase constants from the linear dispersion law. The basic interactions according to eqn. (10) are three-wave $(n = 3)$ and four-wave $(n = 4)$ mixing in case of quadratic and cubic non-linearity, respectively. The three-wave processes include the well-known travelling-wave frequency up- or down-conversion, parametric amplification and SHG where, usually, steady-state amplitudes are assumed. Taking explicitly time-dependent amplitudes into account the three-wave processes can be applied to convolution and correlation techniques as has been discussed in the area of non-linear acoustic waves by Quate and Thompson (1970) and Luukkala and Kino (1971).

To get an insight into the influence of dispersion it is advantageous to study the exemplary case of lossless SHG $(\omega_2 = 2\omega_1)$. The ansatz of two interacting harmonics yields the rate equations $(g(V) = \delta V)$

$$j\left(\tilde{V}_{1x} + \frac{1}{u_{\delta 1}} \tilde{V}_{1t}\right) = -\frac{4}{3} \beta_1 \delta(2V_0 - V_\omega)\tilde{V}_1 + \tilde{V}_1^* \tilde{V}_2 \exp (-j\Delta \beta x)$$

(11)

$$j\left(\tilde{V}_{2x} + \frac{1}{u_{\delta 2}} \tilde{V}_{2t}\right) = -\frac{4}{3} \beta_2 \delta(4V_0 - V_\omega)\tilde{V}_2 + \tilde{V}_1^2 \exp (j\Delta \beta x)$$

(12)

$$V_0 = \frac{4}{3} \delta \left\{ |\tilde{V}_1|^2 + |\tilde{V}_2|^2 \right\} (1 - \delta V_0)^{-1}$$

(13)

where the subscripts $x$ and $t$ denote partial differentiations, $u_{\delta} = \partial \omega / \partial \beta$ is the group velocity and $\Delta \beta = \beta_2 - 2\beta_1$ is a measure of dispersion. The steady-state solution of (11) to (13) gives the spatial variation of the amplitude of the second harmonic, normalized to input amplitude

$$\frac{\tilde{V}_2(x)}{\tilde{V}_1(0)} = \left(\frac{\beta_2}{2\beta_1}\right)^{1/2} v_b \text{sn} \left\{\frac{v_c}{4} \delta \tilde{V}_1(0) (2\beta_1 \beta_2)^{1/2} x ; \frac{v_b}{v_c}\right\}$$

(14)

where $v_b$ and $v_c$ are evaluated from

$$(v_{b,c})^2 = 1 + \frac{\Delta s^2}{8} + \frac{\Delta s}{2} \left(1 + \frac{\Delta s^2}{16}\right)^{1/2}$$

(15)

Here, $\Delta s$ is the following non-linear index or similarity parameter (cf. Armstrong et al. (1962), Wedding and Jäger (1982 a)):

$$\Delta s = \frac{4 \Delta \beta}{\delta \tilde{V}_1(0)} \left(\frac{2}{\beta_1 \beta_2}\right)^{1/2}$$

(16)
‘sn’ denotes the Jacobi elliptical sine function (Jahnke et al. 1960). As a first result, eqn. (14) leads to the well-known interference structure

$$\tilde{\Phi}_2(x) = \frac{\beta_2}{4\Delta \beta} \delta \tilde{\Phi}_1(0) \sin \left( \frac{1}{2} \Delta \beta x \right)$$

(17)

provided that $\Delta s$ is large. In this case of strong dispersion, $\tilde{\Phi}_2(x)$ is periodic and the conversion efficiency is small. Secondly, the most interesting behaviour is concluded from eqn. (14) in case of $\Delta s = 0$. Then the ‘sn’ function is replaced by a ‘tanh’ function and the power of the fundamental wave is totally converted into second-harmonic power. The important point is now to establish phase-matching with $\Delta \beta = 0$ without losing the low-pass filter characteristic.

Wedding and Jäger (1981) have shown how a periodic non-linearity can yield parametric mixing with high efficiencies. The principle is sketched in Fig. 5. Using a periodically doped semiconductor and a periodic width of the centre conductor, a structure with alternating linear and non-linear capacitances arises. The second harmonic is now a backward wave and the output power is now available at the input. The efficiency $\eta$, the ratio of second-harmonic to fundamental power can be estimated from (Fig. 6)

$$8 \arcsin \frac{\sqrt{\eta}}{\sqrt{1-\eta}} = \delta \tilde{\Phi}_1(0) N \sqrt{[B_1 (\pi - B_1)]}$$

(18)

where $N$ is the number of sections and $B_1$ is the phase shift per section. Experiments performed on a transmission line with 14 sections and an open end, revealed a value of $\eta = 95\%$. Parametric amplification has also been measured and a gain of nearly 30 dB has been observed (Wedding and Jäger 1981). In general, these experiments have confirmed the validity of the underlying concept based upon the influence of a low-pass filter structure and the contribution of attenuation in a real slow-wave circuit. So the realization of a device as outlined in Fig. 5 seems to be imaginable, leading to an integrated structure for SHG with a high efficiency or to a parametric amplifier with a large gain.

![Figure 5](image_url)

Figure 5. Schematic top view of the proposed structure for high efficiency SHG (cf. Figs. 2(a) and 2(b)). The elements of an equivalent ladder line are $LCLC_0$ where $C$ is a non-linear and $C_0$ a linear capacitance.
Taking another interesting case, the ansatz in (10) with $i = 1, 3$ and $\omega_3 = 3\omega_1$ implies a third-harmonic generation (THG) process when a balanced structure (Fig. 1(c)) is utilized. Now the rate equations read ($g(V) = \delta' V^2$)

$$j\left(\frac{1}{\xi_{k1}} \frac{1}{\xi_{k3}} \vec{V}_{1z} + \frac{1}{\xi_{k3}} \vec{V}_{3x}\right) = -\frac{1}{2\xi_{k1}} \delta' \left[3\vec{V}_{1z}\left(|\vec{V}_{1z}|^2 + 2|\vec{V}_{3x}|^2\right) + 3\vec{V}_{1z}^* \vec{V}_{3x} \exp(-j\Delta\beta' x)\right]$$

(19)

$$j\left(\frac{1}{\xi_{k3}} \vec{V}_{3x} + \frac{1}{\xi_{k3}} \vec{V}_{3z}\right) = -\frac{1}{2\xi_{k3}} \delta' \left[3\vec{V}_{3x}\left(|\vec{V}_{3x}|^2 + 2|\vec{V}_{3z}|^2\right) + 3\vec{V}_{3x}^* \vec{V}_{3z} \exp(j\Delta\beta' x)\right]$$

(20)

where the dispersion is now expressed by $\Delta\beta' = \beta_3 - 3\beta_1$. Due to the contributions from the first terms on the RHS of (19) and (20) no analytical steady-state solution can be formulated now. Further on, no case of perfect phase-matching exists using reasonable physical parameters, $\vec{V}_3(x)$ being always a periodic function (Armstrong et al. 1962). However, the maximum power efficiency at half of this period can be obtained from (Fig. 7)

$$\left(\frac{3}{2s} - 6s + \frac{3}{2}s^3\right)\eta - \frac{3}{s} + 6s \pm \sqrt{(1 - \eta)^3} = \Delta s'$$

(21)

where $\eta$ is the minimum value satisfying eqn. (21). The parameter of relative dispersion is given by $s^2 = 1 + \Delta\beta'/3\beta_1$ and the non-linear index is now

$$\Delta s' = \frac{8\Delta\beta'}{\delta' \vec{V}_{1z}^2(0)} \left(\frac{3}{\beta_1 \beta_3}\right)^{1/2}$$

(22)

As can be seen, (i) even under phase-matching conditions, $\Delta\beta' = 0$, the efficiency is only $\eta = 4/13$, i.e. the power conversion is incomplete in contrast to SHG; (ii) on the other hand, $\eta$ can be as high as 0.85 provided that non-linearity and dispersion are
well selected, for example $s^2 = 1.05$ and $\Delta s' = 0.8$. The physical interpretation of this feature is as follows. Equations (19) and (20) give rise to self-phase modulation (§ 8) and this non-linear phase-mismatch must partially be compensated by a proper linear dispersion $\Delta \beta' = 0$ in order to enhance the efficiency.

7. Stationary waves and solitons

To introduce the powerful concept of stationary waves, consider the special case of $\bar{V}_j = \text{const.}$ in eqns. (11) and (12) or similarly in (19) and (20). In doing so one immediately gets a stationary solution where the amplitudes of the harmonics are fixed and the phase velocities are equal. What is the meaning of this situation? Such a wave is stationary as a result of non-linear dispersion, i.e. due to the non-linearity the dispersion law is altered in such a way that the linear dispersion $\Delta \beta$ or $\Delta \beta'$ is exactly compensated to yield identical phase velocities. In an SHG measurement this situation can occur as a result of the inevitable losses leading to a stationary waveform $V(x, t)$ which can be observed experimentally. Basically, stationary waves of this kind are obtained from a suitable \textit{ansatz} in eqn. (10). It is, however, easier to discuss this concept explicitly in the time domain.

The dynamic steady-state solutions of a given wave equation are calculated by setting

$$V(x, t) = V(x - u_s t)$$  \hspace{1cm} (23)

Then the partial derivatives are related by

$$\frac{\partial}{\partial x} = -\frac{1}{u_s} \frac{\partial}{\partial t}$$

and the original partial differential equation is reduced to an ordinary differential equation to be solved by common analytical methods. $u_s$ plays the role of an unknown parameter, the velocity of the wave under examination. It should be noted
at this point that due to dissipation in a real system, no stationary waveform can exist in the strong sense of the definition. Nevertheless, since the attenuation is of second order the interplay between non-linearity and dispersion will lead to stationary waves which in turn undergo a further weak dissipation.

Here the stationary waves of eqn. (1) are of interest. Beyond periodic waves as mentioned above, the pulse- or step-like waveforms, i.e. the bounded solitary waves are discussed. Neglecting attenuation the solutions are (for KdV, see Jäger (1982))

\[
V(x, t) = \begin{cases} 
\cosh^{-2} \left\{ \frac{\delta^2}{12 \kappa} \right\}^{1/2} \left\{ \frac{x}{u_0} \left( 1 - \frac{\delta^2}{6} \right) - t \right\} & \text{for KdV} \\
\pm \cosh^{-1} \left\{ \frac{\delta^2}{24 \kappa} \right\}^{1/2} \left\{ \frac{x}{u_0} \left( 1 - \frac{\delta^2}{12} \right) - t \right\} & \text{for MKdV} \\
\pm \tanh \left\{ \frac{-\delta^2}{6 \kappa} \right\}^{1/2} \left\{ \frac{x}{u_0} \left( 1 - \frac{\delta^2}{12} \right) - t \right\} & \text{for nMKdV}
\end{cases}
\]

The pulse-like solutions are also called 'solitons' because they keep their identity even during mutual interactions like overtaking or collision. The step-like waveforms of the nMKdV system in (26) are called 'kinks'.

To elucidate the significance of the concept of stationary waves, Fig. 8 shows experimental results of waveforms observed on transmission lines as sketched in Figs. 3 (c) to (e). These detected waveforms can be interpreted as follows. Figure 8 (a) reveals the generation of two positive solitons per period of the sinusoidal input

Figure 8. Experimental observation of stationary waveforms from a sinusoidal input signal, \( S \) denotes a soliton and \( K \) a kink: (a) KdV, (b) MKdV and (c) nMKdV system.
signal. The voltage wave in Fig. 8(b) exhibits two positive and two negative solitons whereas in Fig. 8(c) solitary kinks are generated superimposed by two further solitons moving on the top and on the bottom. Here the symmetry is broken so that these solitons are KdV solitons. In all cases it has been verified that those individual parts of the waveform are stationary, although the whole wave changes during propagation because it consists of several solitary pulses or steps moving with different velocities. This is the concept of stationary waveforms: Any arbitrary initial signal decomposes into a well-defined superposition (non-linear) of individual solitary waveforms; then the propagation of the wave in the time domain can be regarded as a permanent interaction of stationary pulses and shock waves where the individual velocity depends upon the amplitude as given by eqns. (24) to (26).

In the past, two proposals have been made for application of electrical solitons to communication systems. Suzuki et al. (1973) investigate a secure system where interacting, phase-modulated soliton trains carry the information to be transmitted. Chu and Whitbread (1978) report on a pulse-coded modulation (PCM) communication system where solitons are the signal carriers in a dispersive channel.

8. Self-phase modulation

As a result of interacting harmonic waves, but even in the case where only the fundamental wave with angular frequency \( \omega_0 \) travels down the line, the phase velocity depends on the amplitude of the sine wave. As a further peculiarity, a modulation instability can occur and a modulated wave is changed due to mixing of the sidebands and group velocity dispersion. Such non-linear envelope waves are discussed in this section.

In case of a quadratic non-linearity the first term on the right-hand side sum in eqn. (1i) describes the self-phase modulation. The corresponding non-linear phase constant reads (Wedding and Jäger 1982 b, Wedding et al. 1983)

\[
\beta_{nl} = \beta (1 - \delta^2 \bar{V}^2 / 8)
\]  

(27)

where \( \bar{V} \) is the amplitude of the sinusoidal wave (\( \bar{V} = \bar{V}_1, \bar{V}_2 = 0 \)). As can be seen, this contribution is a direct result of the bias voltage \( V_0 \) generated by the non-linear capacitor. Obviously, the generation process is a transient, the dynamics of eqn. (13) being rewritten as (Figs. 3 (b) to (e))

\[
\tau_1 \frac{d}{dt} \left\{ V_0 + \tau_2 \frac{dV_0}{dt} - \frac{1}{4} \delta \bar{V}^2 \right\} = V_\omega = - \left\{ V_0 + \tau_2 \frac{dV_0}{dt} \right\}
\]  

(28)

where two relaxation mechanisms are involved according to \( \tau_1 = R_0 C'_0 \) and \( \tau_2 = C'_0 / G' \). If \( \tau_1 \to \infty \), eqn. (28) reduces to the well known Debye relaxation. In the slow-mode region, however, this relaxation can generally be ignored, i.e. \( \tau_2 \to 0 \). Equation (28) takes into account that the bias voltage is only present if \( \bar{V} \) changes with time. In the stationary regime \( V_0 = V_\omega \).

The cubic non-linearity on the other side also leads to a non-linear phase constant caused by self-interaction. The following expression is derived from eqn. (19) and \( \bar{V}_3 = 0 \)

\[
\beta_{nl} = \beta (1 - \delta^3 \bar{V}^2 / 8)
\]  

(29)

Thus in case of cubic non-linearity the self-phase modulation is always present and phase-matched and no relaxation takes place.
Equations (27) and (29) can be evaluated much more simply by considering a non-linear capacitance according to (6) and (7), respectively, and an impressed sinusoidal AC voltage. An effective capacitance can be estimated which depends on the voltage amplitude and a non-linear phase constant is obtained. It should be noted that the contribution of harmonic waves also yield a non-linear dispersion as can be gathered from (11) and (19). These contributions can be neglected in the frequency region of strong dispersion. In the following, eqn. (29) is taken as an example since the relation in eqn. (27) is similar. The corresponding effective capacitance should be written as

\[ C'(V) = C_0 (1 - \delta' \dot{V}^2 / 4) \]  

(30)

In the field of non-linear optics \( C' \) resembles the dielectric constant and the non-linear dispersion is known as a refractive index which depends on light intensity.

In a first step, a wave packet with amplitude \( \hat{V}(t) \) propagating down that line experiences a self-phase modulation or an AM–PM conversion. The corresponding frequency chirp at \( x = x_0 \) is

\[ \omega(t) = \omega_0 \left[ 1 + x_0 \frac{\delta'}{8u_0} \frac{\partial \dot{V}^2}{\partial t} \right] \]  

(31)

and the sign of the chirp depends on that of \( \delta' \). The next stage of wave evolution is the distortion of the envelope. Self-steepening and shock wave formation are expected to be limited by group velocity dispersion. A complete description is obtained from the ansatz (10) ignoring higher harmonics and \( V_0 \). Assuming cubic non-linearity, \( g(V) = \delta' V^2 \), the wave equation now yields (cf. eqn. (19))

\[ \ddot{V}_x + \frac{1}{u_k} \dot{V}_t - \frac{\delta' \omega_0}{4u_0} \left( j |\dot{V}|^2 \dot{V} + 2 \frac{\omega_0}{\omega_0} (|\dot{V}|^2 \dot{V}) \right)_t = 0 \]  

(32)

Here \( u_k = \partial^2 / \partial \omega^2 \) denote group velocity dispersion. Equation (32) is a modified non-linear Schrödinger (MNLS) equation (Mio et al. 1976) showing in turn the influences of dispersion, self-phase modulation and self-steepening.

Setting \( \xi = u_k x/2 \) and \( \eta = t - x/u_k \) and neglecting self-steepening which obviously is of second order the following non-linear Schrödinger equation is obtained

\[ \ddot{V}_\xi - j \dot{V}_\eta - j \sigma |\dot{V}|^2 \dot{V} = 0 \]  

(33)

where the similarity parameter \( \sigma = \delta' \omega_0/(4u_k u'_k) \) has been introduced. In case of the cubic dispersion law in (2) one recognizes \( u'_k = 3 \sigma \omega_0 / u_0 \).

A first property can be seen from the dispersion relation of a small-signal sinusoidal envelope wave with frequency \( \Omega \) and phase constant \( K \) superimposed on the carrier with amplitude \( \hat{V}_0 \). The wave equation (32) for the complex envelope yields

\[ K = \frac{\Omega}{u_k} \left[ 1 - \frac{1}{2} \delta' \hat{V}_0^2 \pm \frac{1}{2} \left( (\delta' \hat{V}_0^2 - 2 \omega_0 u_k u'_k \delta' \hat{V}_0^2 + 4(u_k u'_k \Omega^2)^2)^{1/2} \right) \right] \]  

(34)

provided that the non-linearity is not too large. The ± signs are valid if \( u'_k \geq 0 \). Equation (34) predicts an instability of the wave if \( u'_k \delta' > 0 \)—as realized in the MKdV system—and the dispersion is high enough. In other words, an amplification of the sideband amplitudes occur (Yagi and Noguchi 1976).
The non-linearity causing self-phase modulation can be balanced by the group velocity dispersion to yield stationary envelope solitons from eqn. (33). The MKdV system with \( u_g \delta' > 0 \) exhibits a bright soliton of the form (cf. eqn. (25))

\[
\hat{\psi}(x, t) = \hat{\psi}_s \cosh^{-1} \left[ \sqrt{\left( \frac{\delta' \rho^2}{8u_g u_g'} \right)(\frac{x}{u_s} - t)} \right]
\]

where the velocity weakly depends on the amplitude \( \hat{\psi}_s \) according to

\[
u_s = u_g(1 + \sqrt{8\rho^2 (\delta' u_g'/8)\delta'}^2)
\]

These envelope solitons are presently proposed to be used as signal carriers in optical monomode fibres using near-infrared laser light (Hasegawa and Kodama 1981).

The pulse-narrowing due to the combined effects of non-linearity and dispersion can be useful in pulse compression techniques. Figure 9 elucidates the typical behaviour of the nMKdV and the MKdV system with different signs of \( \delta' u_g' \), using a gaussian initial pulse at \( x = 0 \) with fixed carrier frequency \( \omega_0 \). A broadening of the pulse during transmission occurs in the nMKdV system (Fig. 9(a)) and a positive-frequency chirp is established within the middle part which can basically be used for compression in a subsequent dispersive medium. Nevertheless, a much more interesting feature is observed if the dispersion of the transmission line itself can be used as outlined in Fig. 9(b). Now the combined effects of self-phase modulation and dispersion lead to a clear pulse compression with enhanced peak power. In order to give a simple description of the limits of this process one can imagine that the final pulse waveform is just the stationary soliton of eqn. (35). Assuming conservation of pulse energy, the ultimate peak power can be estimated from input energy and final pulse width. Quite the same mechanism is presently used to get ultra-short laser pulses in the femtosecond time scale by using an optical fibre (Mollenauer and Stolen 1984).

![Figure 9](image-url)  
(a)  
(b)

Figure 9. Numerical results of pulse propagation as given by (32). (a) Pulse broadening and generation of a positive-frequency chirp, (b) pulse compression and generation of a negative-frequency chirp.
9. Non-linear resonator

There are several reasons to finally study the characteristics of a non-linear transmission line of finite length with distinct reflections at the ends, i.e. the behaviour of a non-linear resonator. A first reason is concerned with the general behaviour of a periodic structure as sketched in Fig. 2(a) if the length of the non-linear section becomes comparable to the wavelength of the slow wave. It is also evident, that the reflections at the input and output connections of a non-linear transmission line can cause resonance effects even in case of impedance-matching of the small-signal fundamental wave. As a result of resonance it is yet expected that non-linear effects are greatly increased. Moreover, one can imagine that the phenomena of a non-linear resonator could also be interesting for special applications.

The fundamental behaviour of a non-linear resonator may be traced back to an enhanced voltage amplitude inside the device at resonance and hence an increased non-linearity. The total interaction length may be increased considerably as a result of multiple forward and backward wave propagation. On the other hand, the study of non-linear waves by taking reflections into account has proven to be extremely difficult. However, some results have been discovered recently again using the experimental slow-wave model line here with a quadratic non-linearity. The reflecting 'mirrors' are provided by a series capacitance, i.e. a gap in the centre conductor (Wedding et al. 1983) or using a ring resonator (Paulus et al. 1984). In the first instance it should be mentioned that, as a rule, the effects of harmonic or sub-harmonic generation, soliton formation, self-phase modulation or pulse compression can now be detected at low input-power levels. A crucial point however, is the length of the resonator.

In a $\lambda/2$-resonator the dependence of wavelength on the amplitude according to eqns. (27) and (29) leads to an asymmetric non-linear resonance curve by shifting the resonance frequency (Wedding and Jäger 1982 b). As a typical result, the output versus input power characteristic exhibits bistability at a fixed frequency (Fig. 10)

![Figure 10](image-url)

Figure 10. Measured output power as a function of input power of a $\lambda/2$ transmission line resonator, frequency is about 2.5% above resonance.
similar to the optical bistability (Abraham and Smith 1982). Here the non-linear dispersion is caused by SHG with an enhanced conversion efficiency due to a resonance behaviour. Under suitable conditions a value of $\eta > 40\%$ has been measured. Such a bistable device can be used as an effective fast switch, a memory or for logical operations. Obviously the resonator can be a sensitive modulator for microwaves. Besides the aforementioned stationary properties there are several interesting transient and dynamic processes. The transient behaviour is characterized by anomalous switching like overshoot and alternate switching; additionally an instability can occur where a CW input signal is converted into a self-pulsing output wave (Wedding et al. 1983) which can be locked by an external generator.

Using a long resonator, $n\lambda/2$ with $n \geq 2$, the device behaves as follows. In the frequency domain, harmonics and subharmonics down to $\omega/n$ can be generated with large amplitudes. All Fourier components are phase-locked. That implies that short pulses are observable in the space and time domain. This kind of mode-locking yields video solitons (§ 7) or envelope solitons (§ 8) propagating in the resonator. Thus this phenomenon resembles that of a mode-locked laser or the soliton laser (Mollenauer and Stolen 1984) where short pulses of extremely high power are generated in a manner of a travelling-wave pulse compression. In the microwave case, the resonance curve, for example, exhibits a typical multistability, i.e. there are more than two values of output power for the same setting of all parameters, see Fig. 11. Here the resonator shows eight different stable states which correspond to several soliton modes within the resonator (Gasch et al. 1984). Further experiments on this point are in progress especially concerning the power efficiency and influence of loss.

Even in the absence of any reflection under small-signal conditions the transition from a linear transmission line to a non-linear slow-wave structure with dispersion can act as a considerable disturbance for wave propagation. This is expected because the characteristic impedance as well as the cutoff frequency depend on the

![Figure 11. Measured non-linear resonance curve of a $7\lambda/2$ transmission line resonator showing multistability.](image-url)
line capacitance which in turn is a function of voltage amplitude, eqn. (30). Bistability and instabilities are imaginable. Consider for example the following situation. If under small-signal conditions the input frequency is chosen to be slightly below cutoff, a high input power can decrease the cutoff frequency so that wave propagation is prevented more and more. Depending on the non-linearity, the opposite case is also possible. Further investigations have been set about this point.

10. Conclusions

As a summary, non-linear waves in periodic slow-wave structures are shown to reveal several interesting phenomena which may be useful in modern microwave communication systems. Although it has been pointed out that such devices can conveniently be realized by using MMIC technology there are other possibilities to achieve the required non-linearity and dispersion, for example by application of a non-linear dielectric material in a suitable waveguide. However, the proposed slow-wave structures where the non-linearity arises from the voltage-dependent depletion-layer width in the semiconducting substrate are today the most promising devices for non-linear wave applications. The basic performances of these circuits seem to be fairly good, nevertheless there is one central drawback limiting the usefulness, namely, the attenuation of the wave. For any practical design, these losses have to be minimized. However, it is worthwhile to note that several devices will only demand for a low value of the quality factor as given by the attenuation per guide wavelength. This point should not be missed in design considerations. The model line has already been used as a guide. Finally, another aspect of technical importance should also be added, that is the possibility of an external optical control of such semiconductor devices and an integration with other optoelectronic elements. It is expected that future investigations will also be dedicated to this technology.

REFERENCES


